Correlated disordered interactions on Potts models

P. T. Muzy,* A. P. Vieira,† and S. R. Salinas ‡
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, Sao Paulo, Brazil
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Using a weak-disorder scheme and real-space renormalization-group techniques, we obtain analytical results for the critical behavior of various q-state Potts models with correlated disordered exchange interactions along d_1 of d spatial dimensions on hierarchical (Migdal-Kadanoff) lattices. Our results indicate qualitative differences between the cases $d-d_1=1$ (for which we find nonphysical random fixed points, suggesting the existence of nonperturbative fixed distributions) and $d-d_1>1$ (for which we do find acceptable *perturbartive* random fixed points), in agreement with previous numerical calculations by Andelman and Aharony [Phys. Rev. B **31**, 4305 (1985)]. We also rederive a criterion for relevance of correlated disorder, which generalizes the usual Harris criterion.

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I. INTRODUCTION

The effects of disorder on the critical properties of statistical models have been the subject of much work in the last decades. In the context of random interactions, Harris [1] derived a heuristic criterion to gauge the relevance of uncorrelated disorder to the critical behavior, which is predicted to remain unchanged if the specific-heat exponent α of the underlying pure system is negative. If $\alpha>0$, disorder becomes relevant and, in the language of the renormalization group (RG), one expects a flow to a new fixed point (characterized by a nonzero-width fixed distribution of the random variables).

It later became clear that the Harris criterion must be generalized in a number of situations [2–6], since α is not always identifiable with ϕ, the crossover exponent of the width of the distribution of the disorder variables. In particular, random variables correlated along d_1 of the d spatial dimensions give rise to the scaling relation [2,4]

$$\phi=\alpha+d_1\nu, \tag{1}$$

where ν is the correlation-length exponent of the pure system. Using a real-space RG approach based on numerical calculations [7], Andelman and Aharony [4] investigated various q-state Potts models with random exchange constants, finding qualitative differences between the cases $d-d_1>1$ (which yields finite-temperature fixed distributions) and $d-d_1=1$ (which embodies the McCoy-Wu model [8] and yields an ''infinite-disorder'' zero-temperature fixed point). An intuitive illustration of the special role of the $d-d_1=1$ case is that, for any infinitesimal concentration of zero bonds (with a suitable assignment of the random interactions), the system would break into noninteracting $(d-1)$-dimensional structures, and the RG flows would be redirected to the pure fixed point of the corresponding system in $d-1$ dimensions.

In the present paper, we use a (perturbative) weak-disorder [9,10] real-space RG scheme to analyze the critical behavior of q-state Potts models with correlated disordered exchange interactions on various hierarchical lattices, whose exact recursion relations are equivalent to those produced by Migdal-Kadanoff approximations for Bravais lattices. Using this weak-disorder scheme, we obtain analytical results by truncating the recursion relations for the moments of the disorder distribution (which are supposed to remain sufficiently small under the RG iterations). All calculations are performed in the vicinity of $\phi=0$, in a region where disorder is relevant. Depending on the diference between the dimensionality of the system (d) and the number of dimensions in which disorder is correlated (d_1), we distinguish two possibilities: (i) For $d-d_1=1$, the weak-disorder scheme produces a nonphysical fixed-point probability distribution, characterized by a negative variance, which suggests the existence of a nonperturbative (''infinite-disorder'') fixed-point; (ii) For $d-d_1>1$, the scheme yields a physically acceptable perturbative fixed-point distribution. Although obtained by an alternative approach, the main results of this paper are in agreement with the numerical findings of Andelman and Aharony [4].

The outline of the paper is as follows. We first rederive Eq. (1), and obtain a criterion for relevance of correlated disorder involving the number of independent random variables in the unit cell of the lattice and the first derivative of the recursion relations at the pure fixed point. This is done in Sec. II. In Sec. III, we consider q-state Potts models on various hierarchical lattices with $d-d_1=1$. Using a weak-disorder scheme, we obtain a new (random) fixed point for q larger than a characteristic value q_0, where disorder becomes relevant. As in a previous publication [10], this fixed point is located in a nonphysical region of the parameter space, suggesting that a nonperturbative fixed point must be present. In Sec. IV we study a similar problem with $d_1=1$ and $d=3$. In this case we obtain a physically acceptable, finite-disorder fixed point, for $q>q_0$, as in the fully disordered model studied by Derrida and Gardner [9] (although in our case the usual Harris criterion is not satisfied). In Sec. V, we consider an Ising model ($q=2$) on a diamond lattice with $b=2$ bonds and l branches (where l, instead of q, is the control param-

*Electronic address: ptmuzy@uol.com.br
†Electronic address: apvieira@if.usp.br
‡Electronic address: ssalinas@if.usp.br

eter), which constitutes another example of a $d-d_1=1$ system. As in Sec. III, weak disorder again predicts a nonphysical random fixed point. In the final section we give some conclusions.

II. CRITERION FOR RELEVANCE OF CORRELATED DISORDER

Following Andelman and Aharony [4], we consider a d-dimensional bond-disordered model in which the disorder variables are correlated along d_1 spatial directions. We assume that, under renormalization with a length rescaling factor b, the model satisfies a recursion relation $R(x_1,x_2,\ldots,x_n)$, connecting $n=b^{d-d_1}$ independent (and identically distributed) random variables to a renormalized variable x'. (In this paper, these variables are related to reduced exchange couplings.) Defining the deviations $\varepsilon_i \equiv x_i - x_c$, where $x_c=R(x_c,x_c,\ldots,x_c)$ is the critical fixed point of the pure system, we expand R in a Taylor series about x_c to write

$$\varepsilon' \equiv x'-x_c=\sum_{i=1}^{n}\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\varepsilon_i+\frac{1}{2}\sum_{i,j=1}^{n}\left.\frac{\partial^2 R}{\partial x_i \partial x_j}\right|_{x_c}\varepsilon_i\varepsilon_j+\cdots, \tag{2}$$

$$\varepsilon'^2=\sum_{i,j=1}^{n}\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\left.\frac{\partial R}{\partial x_j}\right|_{x_c}\varepsilon_i\varepsilon_j+\sum_{i,j,k=1}^{n}\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\left.\frac{\partial^2 R}{\partial x_j \partial x_k}\right|_{x_c}\varepsilon_i\varepsilon_j\varepsilon_k+\cdots, \tag{3}$$

and similarly for the higher powers of ε'. Averaging over the random variables we get

$$\langle\varepsilon'\rangle=\sum_{i=1}^{n}\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\langle\varepsilon\rangle+\frac{1}{2}\sum_{i=1}^{n}\left.\frac{\partial^2 R}{\partial x_i^2}\right|_{x_c}\langle\varepsilon^2\rangle+\sum_{i\neq j}\left.\frac{\partial^2 R}{\partial x_i \partial x_j}\right|_{x_c}\langle\varepsilon\rangle^2+\cdots, \tag{4}$$

$$\langle\varepsilon'^2\rangle=\sum_{i=1}^{n}\left(\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\right)^2\langle\varepsilon^2\rangle+\sum_{i\neq j}\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\left.\frac{\partial R}{\partial x_j}\right|_{x_c}\langle\varepsilon\rangle^2+\cdots, \tag{5}$$

and corresponding expressions for the higher moments of the deviations. Since $\langle\varepsilon\rangle$ is a measure of the distance to the fixed point, it plays the role of temperature. On the other hand, $\langle\varepsilon^2\rangle$ is a measure of the strength of disorder.

The critical behavior of the model is related to the eigenvalues of the matrix

$$M_{rs}=\frac{\partial\langle\varepsilon'^r\rangle}{\partial\langle\varepsilon^s\rangle}, \tag{6}$$

evaluated at the fixed point. It is clear that the set of recursion relations for the moments of the deviations always has a pure fixed point $\langle\varepsilon\rangle=\langle\varepsilon^2\rangle=\cdots=0$. At that point, it can be shown [11] that M_{rs} is a triangular matrix, and that its two largest eigenvalues are given by

$$\Lambda_1=\left.\frac{\partial\langle\varepsilon'\rangle}{\partial\langle\varepsilon\rangle}\right|_{\text{pure}}=\sum_{i=1}^{n}\left.\frac{\partial R}{\partial x_i}\right|_{x_c} \tag{7}$$

and

$$\Lambda_2=\left.\frac{\partial\langle\varepsilon'^2\rangle}{\partial\langle\varepsilon^2\rangle}\right|_{\text{pure}}=\sum_{i=1}^{n}\left(\left.\frac{\partial R}{\partial x_i}\right|_{x_c}\right)^2. \tag{8}$$

Assuming that, for all i and j,

$$\left.\frac{\partial R}{\partial x_i}\right|_{x_c}=\left.\frac{\partial R}{\partial x_j}\right|_{x_c}\equiv w, \tag{9}$$

and invoking the usual scaling hypotheses

$$\Lambda_1=b^{y_t} \quad \text{and} \quad \Lambda_2=\Lambda_1^{\phi}=b^{\phi y_t}, \tag{10}$$

which define the thermal exponent y_t and the crossover exponent ϕ, we get

$$\phi y_t=2y_t-(d-d_1). \tag{11}$$

Then, using the hyperscaling relation

$$\alpha=2-\frac{d}{y_t}=2-\frac{d\ln b}{\ln(nw)}, \tag{12}$$

we obtain

$$\phi=\alpha+\frac{d_1}{y_t}=\frac{d-d_1}{d}\alpha+2\frac{d_1}{d}, \tag{13}$$

which clearly shows that the Harris criterion ($\phi=\alpha>0$) is not satisfied in the presence of correlated disorder. As $1/y_t$ is usually identified with the correlation-length exponent ν, this last result is equivalent to Eq. (1). It also shows that, for $d_1>0$, the crossover expoent is larger than α, which indicates that correlated disorder induces stronger (geometrical) fluctuations than uncorrelated disorder.

The general criterion for relevance of disorder is $\phi>0$, that is,

$$\alpha>-2\frac{d_1}{d-d_1}. \tag{14}$$

From Eqs. (7)–(9), this is equivalent to

$$nw^2>1. \tag{15}$$

This last result was also derived in a different context by Mukherji and Bhattacharjee [5] and generalizes a criterion pointed out by Derrida *et al.* [3].

In the case of the fully disordered system analyzed by Derrida and Gardner [9], for which $d_1=0$, the requirement in Eq. (14) turns out to be equivalent to the usual form of the Harris criterion ($\alpha>0$).

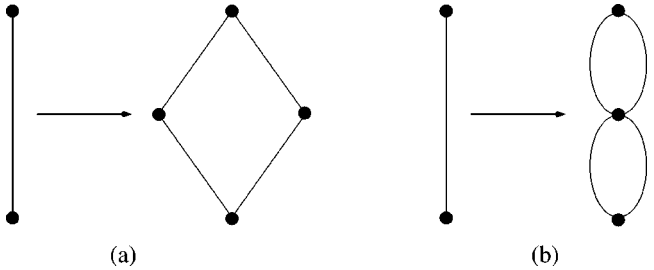

FIG. 1. (a) The diamond hierarchical lattice (with $b=2$ and $l=2$). (b) The necklace hierarchical lattice (with $b=2$ and $l=2$).

III. POTTS MODELS WITH CORRELATED DISORDER: $d-d_1=1$ CASE

The successive generations of a hierarchical lattice are obtained by replacing an existing bond in the previous generation by a unit cell of new bonds in the next generation. In Fig. 1(a), we show the first two stages of the construction of the simple diamond lattice (with $b=2$ bonds and $l=2$ branches). The necklace hierarchical lattice, with $b=2$ bonds and $l=2$ branches, is illustrated in Fig. 1(b).

We now consider a q-state Potts model, given by the Hamiltonian

$$\mathcal{H}_P=-\sum_{(i,j)} J_{ij}\delta_{\sigma_i,\sigma_j}, \tag{16}$$

where the sum is over nearest-neighbor sites on a hierarchical lattice, the spin variables σ_i assume q values, δ is the Kronecker delta symbol, and $\{J_{ij}>0\}$ is a set of independent and identically distributed random variables. Instead of considering a fully disordered arrangement of interactions, we look at correlated disorder, either along layers [see Figs. 2(a) and 2(c)] or along branches [see Figs. 2(b) and 2(d)] of the hierarchical structure.

Introducing the more convenient variable $x_i=\exp(\beta J_i)$, where β is the inverse absolute temperature, it is straightforward to decimate the internal degrees of freedom to obtain

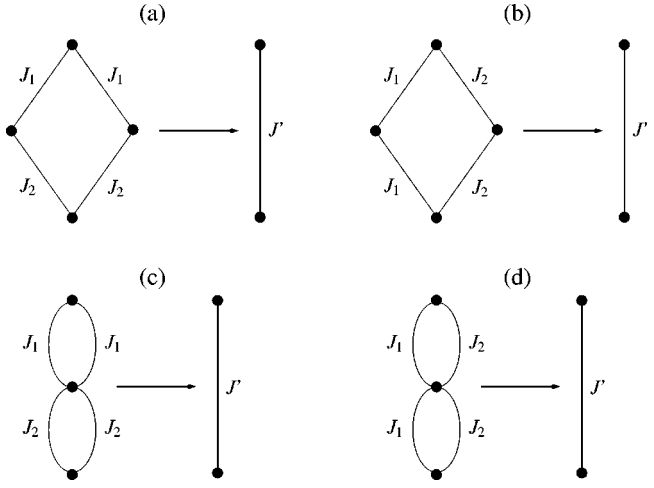

FIG. 2. Correlated distribution of random interactions on diamond and necklace hierarchical lattices.

(Migdal-Kadanoff) recursion relations. In this section we consider the following models.

(A) Random layered diamond lattice, Fig. 2(a), whose recursion relation is

$$x'=R_A(x_1,x_2)=\left(\frac{x_1x_2+q-1}{x_1+x_2+q-2}\right)^2. \tag{17}$$

(B) Random branched diamond lattice, Fig. 2(b), with recursion relation

$$x'=R_B(x_1,x_2)=\left(\frac{x_1^2+q-1}{2x_1+q-2}\right)\left(\frac{x_2^2+q-1}{2x_2+q-2}\right). \tag{18}$$

(C) Random layered necklace lattice, Fig. 2(c), with recursion relation

$$x'=R_C(x_1,x_2)=\frac{x_1^2x_2^2+q-1}{x_1^2+x_2^2+q-2}. \tag{19}$$

(D) Random branched necklace lattice, Fig. 2(d), with recursion relation

$$x'=R_D(x_1,x_2)=\frac{x_1^2x_2^2+q-1}{2x_1x_2+q-2}. \tag{20}$$

Notice that in all these models disorder is correlated along only one spatial direction ($d_1=1$), while the effective dimension is $d=2$. According to Eq. (14), we then expect disorder to be relevant for $\alpha>-2$.

We now write $x'=x_c+\varepsilon'$ and $x_i=x_c+\varepsilon_i$, to perform Taylor series expansions about the critical point of the uniform systems, given by $x_c=R(x_c,x_c)$. For all of these models, with $n=2$ independent values of the exchange parameters (along either layers or bonds), it is straightforward to write the recursion relation

$$\varepsilon'=w(\varepsilon_1+\varepsilon_2)+m(\varepsilon_1^2+\varepsilon_2^2)+f(\varepsilon_1\varepsilon_2^2+\varepsilon_1^2\varepsilon_2)+p\varepsilon_1\varepsilon_2 +c\varepsilon_1^2\varepsilon_2^2+k(\varepsilon_1^3+\varepsilon_2^3)+a(\varepsilon_1^4+\varepsilon_2^4), \tag{21}$$

where w, m, p, f, c, k, and a are model-dependent Taylor coefficients (that depend on the topology of the particular models illustrated in Fig. 2; see Sec. II).

The weak-disorder approximation [9,10] consists in assuming that

$$\langle\varepsilon\rangle\sim\langle\varepsilon^2\rangle\sim\lambda, \tag{22}$$

$$\langle\varepsilon^3\rangle\sim\langle\varepsilon^4\rangle\sim\lambda^2, \tag{23}$$

and in general

$$\langle\varepsilon^{2p-1}\rangle\sim\langle\varepsilon^{2p}\rangle\sim\lambda^p, \tag{24}$$

where $\langle\cdots\rangle$ is a quenched average and λ is a suitable small parameter. Within this approximation, we can use Eq. (21) to write recursion relations for the moments of the deviation, up to second order in λ,

$$\langle \varepsilon' \rangle = 2w\langle \varepsilon \rangle + p\langle \varepsilon \rangle^2 + 2m\langle \varepsilon^2 \rangle + 2f\langle \varepsilon \rangle\langle \varepsilon^2 \rangle + c\langle \varepsilon^2 \rangle^2 + 2k\langle \varepsilon^3 \rangle + 2a\langle \varepsilon^4 \rangle, \tag{25}$$

$$\langle \varepsilon'^2 \rangle = 2w^2\langle \varepsilon \rangle^2 + 2w^2\langle \varepsilon^2 \rangle + 4w(m+p)\langle \varepsilon \rangle\langle \varepsilon^2 \rangle + (2m^2 + 4fw + p^2)\langle \varepsilon^2 \rangle^2 + 4wm\langle \varepsilon^3 \rangle + (4wk + 2m^2)\langle \varepsilon^4 \rangle, \tag{26}$$

$$\langle \varepsilon'^3 \rangle = 3w\langle \varepsilon \rangle\langle \varepsilon^2 \rangle + 3(m+p)\langle \varepsilon^2 \rangle^2 + w\langle \varepsilon^3 \rangle + 3m\langle \varepsilon^4 \rangle, \tag{27}$$

and

$$\langle \varepsilon'^4 \rangle = 3w^2\langle \varepsilon^2 \rangle^2 + w^2\langle \varepsilon^4 \rangle. \tag{28}$$

It is easy to see that there is always a nonrandom fixed point,

$$\langle \varepsilon \rangle = \langle \varepsilon^2 \rangle = \langle \varepsilon^3 \rangle = \langle \varepsilon^4 \rangle = 0, \tag{29}$$

associated with the critical behavior of the pure model. As we pointed out in the previous section, this fixed point becomes unstable with respect to disorder for $2w^2 > 1$. This can also be seen by an inspection of the asymptotic behavior of Eq. (26), which shows that, up to order λ, the renormalized second moment depends only on $\langle \varepsilon^2 \rangle$, with the coefficient $2w^2$. Thus, we expect the onset of a random fixed point at a critical value q_0 of the number of Potts states. From the expression

$$x_c = R(x_c, x_c) \tag{30}$$

for the pure fixed point, we can express q as a function of x_c and, using the condition $2w^2 = 1$, determine the critical value $x_c(q_0)$. For both diamond structures displayed in Figs. 2(a) and 2(b), we have

$$q = (\sqrt{x_c} - 1)(x_c - 1), \tag{31}$$

and $x_c(q_0) = 2.151\,27\ldots$, which leads to $q_0 = 0.537\,32\ldots$. For both necklace structures in Figs. 2(c) and 2(d), we have

$$q = (x_c - 1)(x_c^2 - 1), \tag{32}$$

with $x_c(q_0) = 1.466\,72\ldots$, which also leads to $q_0 = 0.537\,32\ldots$. Disorder is predicted to be relevant for $q > q_0$.

We now introduce the small parameter

$$\lambda = x_c(q) - x_c(q_0) \simeq \left.\frac{dx_c}{dq}\right|_{q_0} (q - q_0) \equiv \left.\frac{dx_c}{dq}\right|_{q_0} \Delta q, \tag{33}$$

to investigate a q-state Potts model in the immediate vicinity of the characteristic value q_0. It should be pointed out that, as the symmetry of the order parameter is one of the factors expected to determine the universality class of the models, Δq is the appropriate parameter to consider. However, λ is more convenient for the algebraic manipulations. From inspection of Eqs. (25)–(28), we see that, up to first-order terms in λ, coefficients w and m are written as

TABLE I. Coefficients of the weak-disorder expansion for the models in Fig. 2.

Coefficient	Model (A)	Model (B)	Model (C)	Model (D)
a	−0.00926	0.00917	−0.92623	0.02894
c	0.08549	0.00016	1.38173	0.07163
k	0.04676	−0.01302	0.25648	−0.02801
f	−0.05370	0.00608	−0.33156	−0.04706
p	0.65117	0.23242	1.56929	0.53634

$$w = \frac{1}{2}\sqrt{2} + w_1\lambda \quad \text{and} \quad m = m_0 + m_1\lambda. \tag{34}$$

It is straightforward to calculate $w_1 = 0.133\,25\ldots$, for the diamond structures, and $w_1 = 0.390\,88\ldots$, for the necklace structures. Also, we have $m_0 = -0.190\,88\ldots$ and $m_1 = 0.198\,65\ldots$, for model (A); $m_0 = 0.018\,49\ldots$ and $m_1 = 0.007\,58\ldots$, for model (B); $m_0 = -0.489\,35\ldots$ and $m_1 = 1.224\,33\ldots$, for model (C); and $m_0 = 0.027\,11\ldots$ and $m_1 = 0.020\,27\ldots$, for model (D). In order to obtain the remaining coefficients, it is enough to keep the zeroth order term in λ (see the values, up to five digits, in Table I).

We are finally prepared to obtain, up to lowest order in Δq, the nonzero values of the moments at the random fixed point. By substituting the weak-disorder assumptions, Eqs. (22) and (23), into Eqs. (25)–(28), and then imposing consistency between equal powers of Δq, we obtain the leading terms for fixed values of the moments as listed in Table II.

In order to perform a linear stability analysis about the fixed points, we have to calculate the eigenvalues Λ_1 to Λ_4 of the matrix

$$M_{rs} = \frac{\partial\langle \varepsilon'^r \rangle}{\partial\langle \varepsilon^s \rangle}.$$

As it should be anticipated from universality, it turns out that the eigenvalues (and so the critical exponents) are the same for models (A) to (D). We always have two eigenvalues, Λ_3 and Λ_4, whose absolute values are smaller than unity. About the pure fixed point, we have

$$\Lambda_1^{(p)} = \sqrt{2} + 0.310\,18\Delta q, \tag{35}$$

$$\Lambda_2^{(p)} = 1 + 0.438\,66\Delta q, \tag{36}$$

with a specific heat exponent

TABLE II. Moments of the deviations defining the random fixed points of the models in Fig. 2 according to the weak-disorder expansion.

Moment	Model (A)	Model (B)	Model (C)	Model (D)
$\langle \varepsilon \rangle / \Delta q$	−14.904	1.0208	−4.4401	0.34798
$\langle \varepsilon^2 \rangle / \Delta q$	−16.170	−11.434	−1.8791	−2.6575
$\langle \varepsilon^3 \rangle / (\Delta q)^2$	1444.5	325.73	46.390	39.946
$\langle \varepsilon^4 \rangle / (\Delta q)^2$	784.41	392.21	10.593	21.187

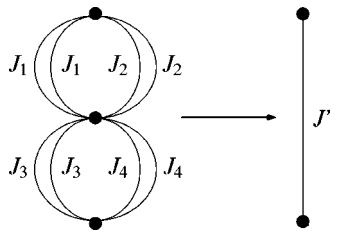

FIG. 3. The hierarchical lattice with $d=3$ and $d_1=1$ considered in Sec. IV

$$\alpha_p = -2 + 2.531\,41\Delta q.$$

At the random fixed point we have

$$\Lambda_1^{(r)} = \sqrt{2} + 0.836\,70\Delta q, \tag{37}$$

$$\Lambda_2^{(r)} = 1 - 0.438\,66\Delta q, \tag{38}$$

which lead to the exponent

$$\alpha_r = -2 + 6.828\,43\Delta q. \tag{39}$$

From Eq. (36), we see that disorder becomes relevant for $\Delta q > 0$. Thus, as shown in Table II, the weak-disorder expansion gives negative (and thus nonphysical) values of the second moment at the random fixed point for models (A) to (D). This suggests that the random fixed point in these systems (for which $d - d_1 = 1$) is nonperturbative, in agreement with numerical calculations [4] that predict an infinite-disorder fixed point. Another odd feature of the weak-disorder results is that the predicted value of the specific-heat exponent in the presence of disorder (α_r) is *larger* than the corresponding quantity (α_p) for the pure model, in disagreement with the general belief that disorder should weaken the transition.

IV. A POTTS MODEL WITH CORRELATED DISORDER: $d - d_1 > 1$ CASE

In order to examine the $d - d_1 > 1$ case, we now consider a Potts model on a necklace hierarchical lattice [4] shown in Fig. 3, with $d=3$ and $d_1=1$. The unit cell contains $n=4$ independent random variables and, in terms of the variables $x_i \equiv \exp(\beta J_i)$, the recursion relation is given by

$$R(x_1, x_2, x_3, x_4) = \frac{x_1 x_2 x_3 x_4 + q - 1}{x_1 x_2 + x_3 x_4 + q - 2}. \tag{40}$$

Following the same steps as in Sec. III, we have

$$q = (x_c - 1)(x_c^2 - 1), \tag{41}$$

$q_0 = 4 + 2\sqrt{2}$, and $x_c(q_0) = 1 + \sqrt{2}$. Performing again the weak-disorder expansion (and truncation), and taking the average over the disorder variables, we obtain the set of recursion relations

$$\langle \varepsilon' \rangle = 4w\langle \varepsilon \rangle + 2(p_1 + 2p_2)\langle \varepsilon \rangle^2 + 4m\langle \varepsilon^2 \rangle + 4(f_1 + 2f_2)\langle \varepsilon \rangle \times \langle \varepsilon^2 \rangle + 2(c_1 + 2c_2)\langle \varepsilon^2 \rangle^2 + 4k\langle \varepsilon^3 \rangle + 4a\langle \varepsilon^4 \rangle, \tag{42}$$

$$\langle \varepsilon'^2 \rangle = 12w^2\langle \varepsilon \rangle^2 + 4w^2\langle \varepsilon^2 \rangle + 8w(3m + p_1 + 2p_2)\langle \varepsilon \rangle \langle \varepsilon^2 \rangle + [12m^2 + 8w(f_1 + 2f_2) + 2(p_1^2 + 2p_2^2)]\langle \varepsilon^2 \rangle^2 + 8wm\langle \varepsilon^3 \rangle + (8wk + 4m^2)\langle \varepsilon^4 \rangle, \tag{43}$$

$$\langle \varepsilon'^3 \rangle = 9w\langle \varepsilon \rangle \langle \varepsilon^2 \rangle + 3(3m + p_1 + 2p_2)\langle \varepsilon^2 \rangle^2 + w\langle \varepsilon^3 \rangle + 3m\langle \varepsilon^4 \rangle, \tag{44}$$

and

$$\langle \varepsilon'^4 \rangle = 9w^2\langle \varepsilon^2 \rangle^2 + w^2\langle \varepsilon^4 \rangle. \tag{45}$$

It should be noted that, due to the smaller symmetry of the lattice, we now have a larger set of coefficients. Also, notice that in this case q_0 is determined from the condition $4w^2 = 1$. About the critical value q_0, and to leading order in Δq, we have

$$w = \frac{1}{2} + \frac{17\sqrt{2} - 24}{4}\Delta q \tag{46}$$

and

$$m = \frac{\sqrt{2} - 2}{8} + \frac{133 - 94\sqrt{2}}{16}\Delta q. \tag{47}$$

The values for the remaining coefficients are listed in Table III.

The moments of the deviations at the random fixed point are written as

$$\langle \varepsilon \rangle = \frac{1}{7}(5 - 3\sqrt{2})\Delta q,$$

$$\langle \varepsilon^2 \rangle = \frac{1}{7}(4 - \sqrt{2})\Delta q,$$

$$\langle \varepsilon^3 \rangle = \frac{3}{49}(95\sqrt{2} - 128)(\Delta q)^2,$$

$$\langle \varepsilon^4 \rangle = \frac{6}{49}(9 - 4\sqrt{2})(\Delta q)^2. \tag{48}$$

TABLE III. Values of the weak-disorder coefficients for the model in Sec. IV.

p_1	p_2	c_1	c_2	f_1	f_2	k	a
$\frac{3\sqrt{2}}{4} - 1$	$\frac{\sqrt{2}}{2} - 1$	$\frac{109\sqrt{2} - 144}{32}$	$\frac{27\sqrt{2} - 38}{32}$	$\frac{25 - 18\sqrt{2}}{16}$	$\frac{11 - 8\sqrt{2}}{16}$	$\frac{3 - 2\sqrt{2}}{16}$	$\frac{7\sqrt{2} - 10}{64}$

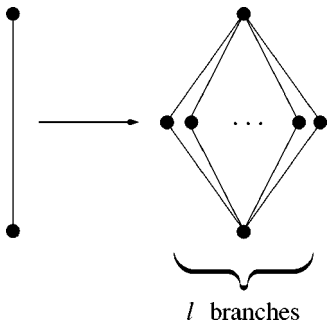

FIG. 4. A diamond hierarchical lattice with $b=2$ bonds and l branches.

Performing a linear stability analysis about the pure fixed point we obtain

$$\Lambda_1^{(p)}=2+(17\sqrt{2}-24)\Delta q, \tag{49}$$

$$\Lambda_2^{(p)}=1+(17\sqrt{2}-24)\Delta q, \tag{50}$$

with a specific-heat exponent

$$\alpha_p=-1+\frac{3}{2}\frac{17\sqrt{2}-24}{\ln 2}\Delta q, \tag{51}$$

while about the random fixed point we have

$$\Lambda_1^{(r)}=2-\frac{1}{7}(92-65\sqrt{2})\Delta q, \tag{52}$$

$$\Lambda_2^{(r)}=1-(17\sqrt{2}-24)\Delta q, \tag{53}$$

with

$$\alpha_r=-1-\frac{3}{14}\frac{92-65\sqrt{2}}{\ln 2}\Delta q. \tag{54}$$

These results show that once more disorder becomes relevant for $\Delta q>0$, but now we obtain a positive (and thus physically acceptable) value of the second moment of the deviations at the random fixed point. We also have $\alpha_r<\alpha_p$. So, as in the fully disordered model ($d_1=0$) studied by Derrida and Gardner [9], and in agreement with numerical calculations [4], the weak-disorder scheme predicts a (perturbative) finite-disorder fixed point, with values of the critical exponents continuously approaching those of the pure model as $\Delta q\to 0$.

V. AN ISING MODEL WITH CORRELATED DISORDER

The set of recursion relations given by Eqs. (25) to (28), with a suitable redefinition of parameters, can also be used to analyze an Ising model on a more general diamond structure with $b=2$ bonds and l branches, and correlated disordered ferromagnetic exchange interactions along the layers (see Fig. 4). For this structure we also have $d-d_1=1$. While in the Potts models we have a natural parameter, q, for varying α, we now change the topology of the lattice, by varying l, to obtain the same effect.

Using the standard Ising Hamiltonian,

$$\mathcal{H}_I=-\sum_{(i,j)} J_{i,j}\sigma_i\sigma_j, \tag{55}$$

with $\sigma_i=\pm 1$, and introducing the more convenient transmissivity variable $t_i=\tanh\beta J_i$, the decimation of the intermediate spins leads to the recursion relation

$$t'=R_l(t_1,t_2)=\tanh\{l\tanh^{-1}(t_1t_2)\}. \tag{56}$$

As in Sec. III, we now write $t'=t_c+\varepsilon$ and $t_i=t_c+\varepsilon_i$, where

$$t_c=R_l(t_c,t_c) \tag{57}$$

is the critical transmissivity of the uniform model. We then perform quenched averages, and use the weak-disorder assumption, to obtain Eqs. (25) to (28).

The critical parameters for relevance of disorder, $l_0=1.449\,76\ldots$ and $t_c(l_0)=0.799\,51\ldots$, come from Eqs. (57) and (15). The small parameter λ can be chosen as

$$\lambda=t_c(l)-t_c(l_0)=\left.\frac{dx_c}{dl}\right|_{l_0}(l-l_0)\equiv\left.\frac{dx_c}{dl}\right|_{l_0}\Delta l. \tag{58}$$

Again we use λ as a convenient parameter for algebraic manipulations, although Δl is the physically relevant variable. The Taylor coefficients in Eqs. (25) to (28) are given by $w=\sqrt{2}/2-0.545\,22\lambda$, $m=-0.496\,98-0.654\,22\lambda$, $a=0.115\,20$, $c=-1.649\,03$, $k=-0.125\,43$, $f=-1.619\,24$, and $p=-0.109\,53$. We then calculate the leading values of the moments at the random fixed point,

$$\langle\varepsilon\rangle=-0.649\,71\Delta l,$$

$$\langle\varepsilon^2\rangle=-0.270\,76\Delta l,$$

$$\langle\varepsilon^3\rangle=-0.300\,84(\Delta l)^2,$$

$$\langle\varepsilon^4\rangle=+0.219\,93(\Delta l)^2. \tag{59}$$

A linear stability analysis leads to the eigenvalues $\Lambda_1^{(p)}=\sqrt{2}+0.718\,84\Delta l$ and $\Lambda_2^{(p)}=1+1.016\,59\Delta l$, for the pure fixed point, and $\Lambda_1^{(r)}=\sqrt{2}+1.205\,37\Delta l$ and $\Lambda_2^{(r)}=1-1.016\,59\Delta l$, for the random fixed point. From these values, we see that disorder is relevant for $\Delta l>0$, but we again have $\langle\varepsilon^2\rangle<0$ in this case.

We then obtain the specific heat critical exponents

$$\alpha_p=-1.071\,63+2.514\,71\Delta l \tag{60}$$

and

$$\alpha_r=-1.071\,63+5.563\,79\Delta l. \tag{61}$$

For $\Delta l<0$, which corresponds to $\alpha<-1.071\,63\ldots$, the pure fixed point is stable and the random model displays the same critical behavior as its pure counterpart. For $\Delta l>0$, which corresponds to $\alpha>-1.0713\ldots$ (yielding again $\alpha_r>\alpha_p$), we anticipate a novel class of (random) critical be-

havior, but the fixed point must be nonperturbative, as suggested by the nonphysical character of the weak-disorder results.

VI. CONCLUSIONS

We have used a weak-disorder scheme and real-space renormalization-group techniques to look at the effects of correlated disorder on the critical behavior of some q-state Potts models with correlated disordered ferromagnetic interactions along d_1 out of d spatial dimensions. We have written exact recursion relations on diamond and necklace hierarchical structures, which are equivalent to Migdal-Kadanoff approximations for the corresponding Bravais lattices.

The weak-disorder scheme leads to analytical results by truncating the recursion relations for the moments of the distribution function. We first used scaling arguments to rederive a general expression for the Harris criterion to gauge the relevance of disorder (and show that it is related to the number of independent random variables in the unit cell of the lattice and the first derivative of the recursion relations at the pure fixed point). We then performed a number of calculations to compare with numerical findings by Andelman and Aharony.

For q-state Potts models on various hierarchical lattices with ferromagnetic random exchange interactions correlated along $d_1=1$ out of $d=2$ dimensions, we obtained a new (random) fixed point for q larger than a characteristic value q_0, where disorder becomes relevant. This fixed point, however, is located in a nonphysical region of parameter space, which suggests that a nonperturbative (infinite-disorder) fixed point must be present (as pointed out by the calculations of Andelman and Aharony). For a q-state Potts model on a diamond lattice with $d_1=1$ and $d=3$, we obtained a physically acceptable, finite-disorder fixed point, for $q>q_0$, as in the fully disordered model analyzed by Derrida and Gardner (although in our case the usual expression of the Harris criterion is not fulfilled). Also, we considered an Ising model ($q=2$) on a diamond lattice with $b=2$ bonds and l branches (where l, instead of q, is the control parameter), which is another example of a $d-d_1=1$ system. Again, the weak-disorder expansion predicts a nonphysical random fixed point.

To summarize the results of this paper, we point out that, in the vicinity of the point where disorder becomes relevant, the weak-disorder scheme always produces a perturbative random fixed point, but there are two distinct possibilities, depending on the difference between d and d_1. (i) If $d-d_1=1$, the perturbative fixed point is characterized by a negative variance, and is thus nonphysical, suggesting the existence of another, nonperturbative fixed point. (ii) If $d-d_1>1$, the scheme predicts a physically acceptable perturbative fixed point. It should be mentioned that this same picture holds for fairly general hierarchical lattices, in particular those with noniterating bonds, as considered by Griffiths and Kauffman [12]. Furthermore, in the case of the quantum Ising model with bond disorder, which corresponds to the extreme-anisotropy limit of the two-dimensional McCoy-Wu model ($d-d_1=1$), Fisher [13] was able to obtain a (presumably exact) fixed-point probability distribution with infinite variance. It is certainly interesting to investigate whether similar conclusions still hold for other models (as the problem of directed polymers in random environments [5]) on either hierarchical or Bravais lattices.

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